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AN ASYMPTOTIC METHOD IN CONTACT PROBLEMS†

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A modification of the "small λ" singular asymptotic method of solving the integral equations of mixed problems in continuum mechanics [1] is proposed in the case of a special behaviour of the symbol of the kernel encountered, for example, in contact problems of the theory of elasticity for cylindrical and conical bodies [2–4]. Contact problems for elastic cylindrical bodies are considered as an example. © 1999 Elsevier Science Ltd. All rights reserved.

Unlike the traditional approach to solving integral equations of the type considered [3, 4], in which the symbol of the kernel is approximated by the sum of two easily factorized functions, which subsequently leads to the need to use a certain iterative process, in this paper the symbol is approximated by a single easily factorized function. This considerably simplifies the solution procedure and enables one to obtain simple approximate formulae for the mechanical characteristics. The possibility of such an approximation was pointed out for the first time in [5].

1. Many mixed problems in continuum mechanics can be reduced to the solution of an integral equation in the function $\varphi(x)$ of the form

$$\int_{-1}^{1} \varphi(\xi) k\left(\frac{x-\xi}{\lambda}\right) d\xi = \pi f(x) \quad (|x| \leq 1) \tag{1.1}$$

$$k(t) = \int_0^\infty K(u) \cos ut\, du \tag{1.2}$$

Here $f(x)$ and $K(u)$ are known functions, where the symbol of the kernel (1.2) $K(u)$ is an even function, meromorphic in the complex plane. Suppose the following expansions exist for $K(u)$

$$\begin{aligned} &K(u) = c_0|u|^{-1} + c_1u^{-2} + c_2u^{-3} + c_3u^{-4} + O(|u|^{-5}), \quad c_0 = 1, \quad (|u| \to \infty) \\ &K(u) = d_0 + d_1|u| + d_2u^2 + d_3|u|^3 + O(u^4), \quad d_0 = A \quad (u \to 0) \end{aligned} \tag{1.3}$$

We will consider two approximations of $K(u)$ by easily factorized functions

$$K^*(u) = \frac{\sqrt{u^2+B^2}}{u^2+C^2} \exp\left(\frac{D}{\sqrt{u^2+E^2}}\right), \quad \frac{B}{C^2}\exp\left(\frac{D}{E}\right) = A, \quad D = c_1 \tag{1.4}$$

$$K^*(u) = \frac{\sqrt{u^2+B^2}}{u^2+C^2} \exp\left(\frac{D|u|}{u^2+E^2}\right), \quad \frac{B}{C^2} = A, \quad D = c_1 \tag{1.5}$$

The positive constants B, C, D and E in these approximations must be chosen so that the function $K^*(u)$ possibly more accurately reflects the behaviour of the function $K(u)$ on the real axis, particularly when $u \to 0$ and $u \to \infty$. Hence, it is recommended that approximation (1.4) should be used if $d_1 = 0$ in (1.3). When $d_1 \neq 0$ (this occurs, for example, in the problem analysed in [5]) one must use approximation (1.5). If the error of approximations (1.4) and (1.5) is too large, they can be made more complicated, without loss of generality, by premultiplying the argument of the exponential function and/or the factor in front of the exponential function by a fraction—the ratio of polynomials of like power in u^2.

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For simplicity we will henceforth confine ourselves to the case when $f(x) = f$. It is well known [6], that the principal term of the asymptotic solution of Eq. (1.1) with a kernel in the form (1.2), (1.4) or (1.5) for small γ can be represented in the form

$$\varphi(x)=\frac{f}{\lambda}\left[\omega\left(\frac{1+x}{\lambda}\right)+\omega\left(\frac{1-x}{\lambda}\right)-\upsilon\left(\frac{x}{\lambda}\right)\right] \tag{1.6}$$

while the functions $\omega(s)$ and $\nu(s)$ are found from the equations

$$\int_0^\infty \omega(\tau)d\tau\int_0^\infty K^*(u)\cos u(\tau-s)du=\pi \quad (0\leqslant s<\infty) \tag{1.7}$$

$$\int_{-\infty}^\infty \upsilon(\tau)d\tau\int_0^\infty K^*(u)\cos u(\tau-s)du=\pi \quad (-\infty< s<\infty) \tag{1.8}$$

We will first consider approximation (1.4). Integral equation (1.8) is solved using the convolution theorem for a Fourier transformation. We thereby obtain $\upsilon(s) = A^{-1}$. To solve integral equation (1.7) we use the Wiener–Hopf method [7]. We finally obtain

$$\omega(s)=\frac{1}{2\pi}\int_\Gamma \frac{e^{-ius}ds}{(-iu)K_+^*(0)K_+^*(u)} \tag{1.9}$$

where the contour Γ is a straight lying slightly above the real axis in the plane of the complex variable u, while the functions $K_\pm^*(u)$ are found by factorization, i.e. the representation

$$K^*(u)=K_+^*(u)K_-^*(u) \tag{1.10}$$

where the function $K_+^*(u)$ $K_-^*(u))$ is regular in the half-plane $\operatorname{Im} u > -\varepsilon$ $(\operatorname{Im} u < \varepsilon)$ and has no zeros here and $\varepsilon = \min(B, C, E)$.

Equation (1.10) is factorized in the form [7]

$$K_\pm^*(u)=\frac{\sqrt{B\mp iu}}{C\mp iu}e^{Df_\pm(u)}$$
$$\frac{1}{\sqrt{u^2+E^2}}=f_+(u)+f_-(u),\quad f_\pm(u)=\frac{\pm i}{\pi\sqrt{u^2+E^2}}\ln\frac{u+\sqrt{u^2+E^2}}{\pm iE} \tag{1.11}$$

Here the functions $f_+(u)$ and $f_-(u)$ are regular in the half-plane $\operatorname{Im} u > -\varepsilon$ and $\operatorname{Im} u < -\varepsilon$, respectively.

We will now obtain that $K_+^*(0) = \sqrt{A}$. For convenience we will take $u = ip$ in (1.9), in which case, using (1.11) we will have

$$\omega(s)=\frac{1}{2\pi i}\int_{\Gamma_*}\frac{\Omega(p)}{p}e^{ps}dp,\quad \Omega(p)=\frac{p+C}{\sqrt{A(p+B)}}e^{g(p)}$$
$$g(p)=\frac{-D}{\pi\sqrt{p^2-E^2}}\ln\frac{p+\sqrt{p^2-E^2}}{E} \tag{1.12}$$

where the contour Γ_* is a straight line lying slightly to the right of the imaginary axis in the plane of the complex variable p.

It can be seen from (1.12) that the function $\Omega(p)$ is a Laplace–Carson transformation of $\omega(s)$. We will henceforth denote this relation as $\Omega(p) : \to \omega(s)$. We will assume that the value of the constant E in (1.4) is fairly large, so that the function $\exp[g(p)]$, where $g(p)$ has the representation (1.12), can be approximated in the half-plane $\operatorname{Re} p > 0$ with a high degree of accuracy as follows:

$$e^{g(p)}\approx 1+g(p) \tag{1.13}$$

Substituting (1.13) into the first relation of (1.12) and using reference tables [8] and the convolution theorem for the Laplace transformation, we obtain the following approximate expression for the function $\omega(s)$

$$\omega(s)=\frac{1}{\sqrt{A}}\left(\frac{e^{-Bs}}{\sqrt{\pi s}}+\frac{C}{\sqrt{B}}\operatorname{erf}\sqrt{Bs}+I(s)\right),\quad I(s)\leftarrow:\frac{p+C}{\sqrt{p+B}}g(p)$$
$$I(s)=\frac{-D}{\pi}\int_0^s\left[\frac{e^{-B(s-\tau)}}{\sqrt{\pi(s-\tau)}}+\frac{C}{\sqrt{B}}\operatorname{erf}\sqrt{B(s-\tau)}\right]K_0(E\tau)d\tau \tag{1.14}$$

Here erf(x) is the probability integral while $K_0(t)$ is a Bessel function, to calculate which one can use special approximations [9].

For the integral characteristic of the solution of Eq. (1.1)

$$P=\int_{-1}^{1}\varphi(x)dx \tag{1.15}$$

using (1.6) we obtain the expression

$$\frac{P}{f}=2\int_0^{\zeta}\omega(\tau)d\tau-\frac{\zeta}{A},\quad \zeta=\frac{2}{\lambda} \tag{1.16}$$

Substituting (1.14) into (1.16) and bearing in mind that [8]

$$\int_0^s I(\tau)d\tau\leftarrow:\frac{p+C}{p\sqrt{p+B}}g(p):\rightarrow J(s)=\frac{-D}{\pi}\int_0^s\left\{\frac{C}{B}\sqrt{\frac{s-\tau}{\pi}}e^{-B(s-\tau)}+\right.$$
$$\left.+\frac{\operatorname{erf}\sqrt{B(s-\tau)}}{\sqrt{B}}\left[1-\frac{C}{2B}+C(s-\tau)\right]\right\}K_0(E\tau)d\tau \tag{1.17}$$

we obtain

$$\frac{P}{f}=\frac{2C}{\sqrt{AB}}\left[\left(\zeta-\frac{1}{2B}+\frac{1}{C}\right)\operatorname{erf}\sqrt{B\zeta}+\sqrt{\frac{\zeta}{\pi B}}e^{-B\zeta}\right]-\frac{\zeta}{A}+\frac{2}{\sqrt{A}}J(\zeta) \tag{1.18}$$

Expression (1.18) can be simplified considerably when $\lambda\leqslant\lambda_*$ if we bear in mind the fact that, from the relation

$$\frac{p+C}{p\sqrt{p+B}}g(p)=\frac{-CD}{2\sqrt{B}Ep}\left[1-\left(\frac{1}{2B}-\frac{1}{C}+\frac{2}{\pi E}\right)p+O\left(p^2\right)\right]\quad(p\rightarrow 0) \tag{1.19}$$

the following asymptotic equality follows [10]

$$J(s)=\frac{-CD}{2\sqrt{B}E}\left(s-\frac{1}{2B}+\frac{1}{C}-\frac{2}{\pi E}+O\left(s^{-1}\right)\right)\quad(s\rightarrow+\infty) \tag{1.20}$$

Substituting (1.2) into (1.18) we obtain, when $\lambda\leqslant 1/4$

$$\frac{P}{f}=\zeta\left[\frac{C}{\sqrt{AB}}\left(2-\frac{D}{E}\right)-\frac{1}{A}\right]+\frac{C}{\sqrt{AB}}\left[\frac{2}{C}-\frac{1}{B}-\frac{D}{E}\left(\frac{1}{C}-\frac{1}{2B}-\frac{2}{\pi E}\right)\right] \tag{1.21}$$

Note that, together with the additive form of the principal term of the asymptotic solution of the problem for small λ (1.6), one can often use the equivalent multiplicative form [6]

$$\varphi(x)=\frac{fA}{\lambda}\omega\left(\frac{1+x}{\lambda}\right)\omega\left(\frac{1-x}{\lambda}\right) \tag{1.22}$$

We will now consider approximation (1.5). The advantage of approximation (1.5) is the factor that using it is much easier to satisfy condition (1.13) than when using (1.4). The non-analyticity of the function $K^*(u)$ of the form (1.5) hinders as effective solution of Eqs (1.7) and (1.8). Hence, we will use the perturbation method, replacing the function $K^*(u)$ by the expression

$$K^*(u)=\frac{\sqrt{u^2+B^2}}{u^2+C^2}\exp\left(\frac{D\sqrt{u^2+\varepsilon^2}}{u^2+E^2}\right) \tag{1.23}$$

and then allowing ε to approach zero.

The solution of integral equation (1.8), after this replacement, is given, as before, by the expression $\upsilon(s)=A^{-1}$ (but here the formula for A is different).

We can easily factorize (1.10) is the argument of the exponential function in formula (1.23) is represented as

$$\frac{D\sqrt{u^2+\varepsilon^2}}{u^2+E^2}=D[g_+(u)+g_-(u)] \tag{1.24}$$

where the functions $g_+(u)$ and $g_-(u)$ are regular in the half-planes $\operatorname{Im} u > -\varepsilon$ and $\operatorname{Im} u < -\varepsilon$, respectively. Then

$$K_\pm^*(u)=\frac{\sqrt{B\mp iu}}{C\mp iu}e^{Dg_\pm(u)} \tag{1.25}$$

To construct the functions $g_\pm(u)$ we use the identity

$$\frac{\sqrt{u^2+\varepsilon^2}}{u^2+E^2}=\frac{1}{\sqrt{u^2+\varepsilon^2}}-\frac{E^2-\varepsilon^2}{2iE}\left[\frac{1}{(u-i\varepsilon)\sqrt{u^2+\varepsilon^2}}-\frac{1}{(u+i\varepsilon)\sqrt{u^2+\varepsilon^2}}\right] \tag{1.26}$$

and the well-known formulae [7]

$$\begin{gathered}\frac{1}{\sqrt{u^2+\varepsilon^2}}=f_+(u)+f_-(u),\quad f_\pm(u)=\frac{\pm i}{\pi\sqrt{u^2+\varepsilon^2}}\ln\frac{u+\sqrt{u^2+\varepsilon^2}}{\pm i\varepsilon}\\ \frac{1}{(u-\xi)\sqrt{u^2+E^2}}=F_+(u)+F_-(u)\\ F_+(u)=\frac{f_+(u)\mp f_\pm(\xi)}{u-\xi},\quad F_-(u)=\frac{f_-(u)\pm f_\pm(\xi)}{u-\xi}\end{gathered} \tag{1.27}$$

where in the last two formulae the upper sign is taken if the point is situated in the upper half-plane and vice versa. From (1.26) and (1.27) we obtain

$$g_+(u)=\frac{1}{\sqrt{u^2+E^2}}\left[\left(u^2+\varepsilon^2\right)f_+(u)-iu\frac{E^2-\varepsilon^2}{E}f_+(iE)\right] \tag{1.28}$$

The expression for $g_-(u)$ is similar.

In (1.28) we take the partial limit as $\varepsilon\to 0$. Then

$$g_+(u)=\frac{-iu}{u^2+E^2}\left[iuf_+(u)+Ef_+(iE)\right] \tag{1.29}$$

It is now easy to see that $K_+^*(0)=\sqrt{A}$. Further, for the function $\omega(s)$ we obtain representation (1.12), where we must put

$$g(p)=\frac{-Dp}{p^2-E^2}[pf(p)-Ef(E)],\quad f(p)=\frac{1}{\pi\sqrt{p^2-\varepsilon^2}}\ln\frac{p+\sqrt{p^2-\varepsilon^2}}{\varepsilon} \tag{1.30}$$

We will write the following chain of relations [8]

$$pf(p):\to\frac{K_0(\varepsilon s)}{\pi}\xrightarrow{\varepsilon\to 0}-\frac{\ln(\varepsilon s/2)+\mathrm{C}}{\pi}\leftarrow:\frac{\ln(2p/\varepsilon)}{\pi} \tag{1.31}$$

where C is Euler's constant.

Using the chain (1.31) we take the final limit as $\varepsilon \to 0$ in expression (1.30) for $g(p)$. We obtain

$$g(p) = \frac{-Dp}{\pi\left(p^2 - E^2\right)} \ln \frac{p}{E} \tag{1.32}$$

The approximate equality (1.30), (1.32), for example, on the positive part of the real p axis is satisfied with high accuracy with less constraints on the value of E from (1.5) (E must not be too small), than in the case (1.4), (1.12) (for the same value of D). Using the first formula of (1.12), (1.13) and (1.32) and reference tables [8], we obtain the following approximate expression for the function $\omega(s)$

$$\omega(s) = \frac{1}{\sqrt{A}}\left(\frac{e^{-Bs}}{\sqrt{\pi s}} + \frac{1}{\sqrt{A}} \operatorname{erf}\sqrt{Bs} + I(s)\right), \quad I(s) \leftarrow: \frac{p+C}{\sqrt{p+B}} g(p)$$

$$I(s) = \frac{D}{2\pi} \int_0^s \left[\frac{e^{-B(s-\tau)}}{\sqrt{\pi(s-\tau)}} + \frac{1}{\sqrt{A}} \operatorname{erf}\sqrt{B(s-\tau)}\right]\left[e^{E\tau}\operatorname{Ei}(-E\tau) + e^{-E\tau}\operatorname{Ei}(E\tau)\right] d\tau \tag{1.33}$$

Here $\operatorname{Ei}(\pm x)$ is the integral exponential function [11], for the numerical realization of which it is convenient to use the formulae [9]

$$\operatorname{Ei}(x) = \ln x + \mathrm{C} + \int_0^x \frac{e^t - 1}{t} dt, \quad \operatorname{Ei}(-x) = -E_1(x) \quad (x > 0) \tag{1.34}$$

where we have the highly accurate approximations 5.1.53 and 5.1.54 [9] for the function $E_1(x)$, while the integral in (1.34) can be evaluated by the Gauss quadrature formula.

As previously, for the integral characteristic Pf^{-1} we obtain the expression ($\zeta = 2/\lambda$)

$$\frac{P}{f} = \left[\frac{2\zeta}{A} - \frac{1}{AB} + \frac{2}{\sqrt{AB}}\right] \operatorname{erf}\sqrt{B\zeta} + \frac{2}{A}\sqrt{\frac{\zeta}{\pi B}} e^{-B\zeta} - \frac{\zeta}{A} + \frac{2}{\sqrt{A}} J(\zeta)$$

$$J(s) = \int_0^s I(\tau) d\tau = \frac{D}{2\pi E} \int_0^s \left[\frac{e^{-B(s-\tau)}}{\sqrt{\pi(s-\tau)}} + \frac{1}{\sqrt{A}} \operatorname{erf}\sqrt{B(s-\tau)}\right] \times \tag{1.35}$$

$$\times \left[e^{E\tau}\operatorname{Ei}(-E\tau) - e^{-E\tau}\operatorname{Ei}(E\tau)\right] d\tau$$

Expression (1.35) can be simplified considerably when $\lambda \leqslant \lambda_*$ if we bear in mind the fact that, from the relation

$$\frac{p+C}{p\sqrt{p+B}} g(p) = \frac{D}{\pi\sqrt{A}E^2}\left[1 + \left(\frac{1}{C} - \frac{1}{2B}\right)p + O\left(p^2\right)\right] \ln \frac{p}{E} \quad (p \to 0) \tag{1.36}$$

we obtain the asymptotic equality [10]

$$J(s) = \frac{-D}{\pi\sqrt{A}E^2}\left[\ln(Es) + \mathrm{C} + \left(\frac{1}{C} - \frac{1}{2B}\right)\frac{1}{s} + O\left(s^{-2}\right)\right] \quad (s \to +\infty) \tag{1.37}$$

Substituting (1.37) into the first formula of (1.35) we obtain, when $\lambda \leqslant 1/4$

$$\frac{P}{f} = \frac{\zeta}{A} - \frac{2D}{\pi A E^2}\left[\ln(E\zeta) + \mathrm{C} + \left(\frac{1}{C} - \frac{1}{2B}\right)\frac{1}{\zeta}\right] + \frac{2}{\sqrt{AB}} - \frac{1}{AB} \tag{1.38}$$

2. As an example, consider the well-known axisymmetric contact problems [2, 3] of the interaction between an infinite elastic cylinder of radius R with a rigid band of width $2a$ and base $r = R - \delta$ (Problem 1), and also the problem of the interaction between an elastic space, weakened by an infinite cylindrical cavity of radius R, with a rigid bearing of width $2a$ and base $r = R + \delta$ (Problem 2). After introducing the dimensionless quantities

$$x = \frac{z}{a}, \quad \lambda = \frac{R}{a}, \quad \varphi(x) = \frac{q(ax)}{\theta}, \quad f = \frac{\delta}{a} \tag{2.1}$$

where $q(z)$ are the unknown normal contact pressures and θ is the contact stiffness, we arrive at Eq. (1.1) in which, for Problem 1

$$K(u)=\left[u^2\left(\Omega_1^2(u)-1\right)-2(1-\nu)\right]^{-1};\quad \Omega_1(u)=I_0(u)/I_1(u) \tag{2.2}$$

while for Problem 2

$$K(u)=\left[u^2\left(\Omega_2^2(u)-1\right)+2(1-\nu)\right]^{-1};\quad \Omega_2(u)=K_0(u)/K_1(u) \tag{2.3}$$

For Problems 1 and 2 in expression (1.3) for large values of $|\mu|$ in the case when $\nu = 0.3$ we must put, respectively

$$\begin{aligned}&1)\ c_1=0.4,\ c_2=-0.965,\ c_3=-2.336\\ &2)\ c_1=-0.4,\ c_2=-0.965,\ c_3=2.336\end{aligned} \tag{2.4}$$

For small values of u we have

$$\begin{aligned}&1)\ K(u)=[2(1+\nu)]^{-1}[1+O(u^4)]\\ &2)\ K(u)=[2(1-\nu)]^{-1}\{1-u^2[2(1-\nu)]^{-1}+O(u^4\ln^2 u)\}\end{aligned} \tag{2.5}$$

It can be seen from (2.5) that approximation (1.4) is more suitable for both problems, in which we must take $E > 1$ to satisfy condition (1.13). Nevertheless, we will also investigate the solution with approximation (1.5), with which comparatively high accuracy can be achieved due to the smallness of the quantity DE^{-2} compared with A, since the term $DE^{-2}|u|$ is present in the expansion of the function (1.5) as $u \to 0$. Further, we will assume that Poisson's ratio $\nu = 0.3$. Then, in the case of (1.4) we have for Problems 1 and 2, respectively

$$\begin{aligned}&1)\ B=1.4442,\ C=2.0929,\ D=0.4,\ E=2.5968\\ &2)\ B=0.8353,\ C=0.8970,\ D=-0.4,\ E=1.0699\end{aligned} \tag{2.6}$$

and the error of approximation (1.4) for all $0 \leqslant u < \infty$ does not exceed 8.7% for Problem 1 and 5% for Problem 2. Moreover, for Problem 1, the asymptotic form (2.5) is taken into account in the accuracy. For the values (2.6) the error of approximation (1.13) and (1.12), for example, on the positive part of the real axis does not exceed 1% for Problem 1 and 2% for Problem 2.

Introducing the values of the constants (2.6) into (1.21), we have

$$1)\ Pf^{-1}=2.5838\,\zeta+0.7882;\quad 2)\ Pf^{-1}=1.3567\,\zeta+1.1648 \tag{2.7}$$

When $\lambda \leqslant 1/4$ the values calculated from (2.7) differ by less than 1% from the corresponding values obtained using (1.18).

Using approximation (1.5) we obtain that

$$\begin{aligned}&1)\ B=1.0962,\ C=1.6882,\ D=0.4,\ E=3.0193\\ &2)\ B=0.3562,\ C=0.7062,\ D=-0.4,\ E=0.7501\end{aligned} \tag{2.8}$$

and the error of approximation (1.5) for all $0 \leqslant u < \infty$ for both problems does not exceed 5%. For the values given by (2.8) the error of approximation (1.13) and (1.32), for example, on the positive part of the real axis for both problems does not exceed 1%.

Introducing the values of the constant (2.8) into (1.38) and dropping terms of the order of ζ^{-1}, in view of their smallness, we have

$$\begin{aligned}&1)\ Pf^{-1}=2.5999\,\zeta-0.07262\ln\zeta+0.5862\\ &2)\ Pf^{-1}=1.4001\,\zeta+0.6337\ln\zeta+0.2181\end{aligned} \tag{2.9}$$

When $\lambda \leqslant 1/4$ the values calculated from (2.9) differ by less than 1% from the corresponding values obtained using (1.35). Moreover, when $\lambda \leqslant 1/4$ the values obtained using (2.9) differ from the values

Table 1

λ	8	4	2	1	½	¼
χ_1^1 [2]	1.44	2.09	3.55	–	–	–
χ_1^1 (1.18)	–	–	3.42	5.97	11.1	21.5
χ_1^1 (1.35)	–	–	3.30	5.77	10.9	21.2
χ_2^1 [2]	0.459	0.685	1.20	–	–	–
χ_2^1 (1.6), (1.4)	–	–	1.29	2.52	5.14	10.3
χ_2^1 (1.6), (1.5)	–	–	1.22	2.42	5.04	10.3
χ_3^1 [2]	0.459	0.651	1.00	–	–	–
χ_3^1 (1.22), (1.4)	–	–	0.896	1.28	1.81	2.57
χ_3^1 (1.22), (1.5)	–	–	0.878	1.27	1.81	2.57
χ_1^2 [2]	1.32	1.77	2.52	–	–	–
χ_1^2 (1.18)	–	–	2.45	3.86	6.59	12.0
χ_1^2 (1.35)	–	–	2.47	3.95	6.87	12.8
χ_2^2 [2]	0.432	0.602	0.924	–	–	–
χ_2^2 (1.6), (1.4)	–	–	0.872	1.48	2.76	5.43
χ_2^2 (1.6), (1.5)	–	–	0.885	1.53	2.92	5.87
χ_3^2 [2]	0.402	0.518	0.640	–	–	–
χ_3^2 (1.22), (1.4)	–	–	0.686	0.937	1.32	1.86
χ_3^2 (1.22), (1.5)	–	–	0.697	0.964	1.37	1.93

obtained using (2.7) by 2% and 7% for Problems 1 and 2, respectively, which confirms the applicability of approximation (1.5) for solving Problems 1 and 2 for small λ.

Table 1 gives values of the quantities

$$\chi_1^n = Pf^{-1},\quad \chi_2^n = \varphi(0)f^{-1},\quad \chi_3^n = \lim \varphi(x)\sqrt{1-x^2}\,f^{-1}\quad (x \to 1) \tag{2.10}$$

calculated using the above formulae for small λ and also using formulae (1.3), (1.12) and (1.14)–(1.18) from [2] for large λ. Note that previously [2, p. 708] the constants c_n (2.4) and, consequently, a_{nm} of the form (2.9) [2] were written incorrectly for Problems 1 and 2 so that the calculations using the "large λ" method from [2] must also be regarded as inaccurate. The recalculated values are as follows:

$$\begin{gathered} 1)\ a_{30} = -0.556,\ a_{20} = -0.628,\ a_{11} = -0.428,\ a_{31} = 1.427.\ a_{21} = -0.612 \\ 2)\ a_{30} = -0.459,\ a_{20} = 0.628,\ a_{11} = -0.428,\ a_{31} = -0.390,\ a_{21} = 0.612 \end{gathered} \tag{2.11}$$

The value of the superscript $n = 1$ and $n = 2$ in (2.10) corresponds to Problems 1 and 2. The values $\chi^n{}_{2,3}$ for $\lambda = 2$, obtained from (1.6) and (1.22), join up with the values of these quantities obtained [3] using another modification of the singular asymptotic "small λ" method. In the neighbourhood of the values $\lambda = 2$ the asymptotic solutions are very close, in the limits of their accuracy. Hence, we can assert that we have obtained an effective solution of Problems 1 and 2 over the whole range of variation of the dimensionless parameter $\lambda \in (0, \infty)$.

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